

$N = 1$  Superstring in  $2 + 2$  DimensionsZ. Khviengia, H. Lu<sup>1</sup>, C.N. Pope<sup>1</sup>, E. Sezgin<sup>2</sup>, X.J. Wang and K.W. Xu

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**Abstract**

In this paper we construct a  $(2, 2)$  dimensional string theory with manifest  $N = 1$  space-time supersymmetry. We use Berkovits' approach of augmenting the spacetime supercoordinates by the conjugate momenta for the fermionic variables. The worldsheet symmetry algebra is a twisted and truncated "small"  $N = 4$  superconformal algebra. The realisation of the symmetry algebra is reducible with an infinite order of reducibility. We study the physical states of the theory by two different methods. In one of them, we identify a subset of irreducible constraints, which is by itself critical. We construct the BRST operator for the irreducible constraints, and study the cohomology and interactions. This method breaks the  $SO(2, 2)$  spacetime symmetry of the original reducible theory. In another approach, we study the theory in a fully covariant manner, which involves the introduction of infinitely many ghosts for ghosts.

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# 1 Introduction

The Green-Schwarz superstring [1], with manifest spacetime supersymmetry, has proved to be notoriously difficult to quantise in a covariant manner. The difficulty stems from the fact that there is no kinetic term for the fermionic spacetime coordinates. This problem has been overcome recently by Berkovits [2] in a reformulation of the superstring, in which the spacetime supercoordinates are augmented by the conjugate momenta for the fermionic variables. The theory has  $N = 2$  worldsheet supersymmetry, as well as manifest four-dimensional  $N = 1$  spacetime supersymmetry. This theory can be thought of as a ten-dimensional theory compactified on a Calabi-Yau background.

It is interesting to investigate whether such an approach could be used for constructing an intrinsically four-dimensional theory with manifest spacetime supersymmetry. This would contrast strikingly with the  $N = 2$  NSR string [3] which, although it has a four-dimensional spacetime (with  $(2, 2)$  signature), has no supersymmetry in spacetime. In fact it has only one Neveu-Schwarz state, describing self-dual Yang-Mills in the open string, and self-dual gravity in the closed string [3], together with an additional Ramond massless state which is also bosonic [4]. Attempts have been made to find a supersymmetric version of the theory. In a recent paper [5], it was observed that massless fermionic physical states, as well as bosonic ones, appear in certain  $Z_2$  twisted sectors of the theory. This is however at the price of breaking the spacetime structure, and in addition it does not have the usual definition of spacetime supersymmetry.

If a four-dimensional string of the Berkovits type could be constructed, it would be quite different from the above case, in that it would have a manifest spacetime supersymmetry. A way to build such a theory is suggested by some work of Siegel [6]. He considered a set of quadratic constraints built from the coordinates and momenta of a superspace in  $2 + 2$  dimensions, and thus displaying manifest spacetime supersymmetry. In [6] it was proposed that in the case of open strings, this theory described self-dual  $N = 4$  super Yang-Mills, whilst the corresponding closed string described self-dual  $N = 8$  supergravity.

The full set of constraints considered in [6] do not generate a closed algebra. However, we find that there exists a subset of the constraints that does close on an algebra, with two bosonic spin-2 generators and two fermionic spin-2 generators. In this paper we build a Berkovits-type open string theory in four dimensions, based on this worldsheet symmetry algebra. Noting that the central charge in the ghost sector vanishes, we see that the matter fields should also have zero central charge. We achieve this by taking the coordinates  $(X^\mu, \theta^\alpha)$  of a chiral  $N = 1$  superspace, together with the canonical momenta  $p_\alpha$  for the fermionic coordinates. This is a chiral restriction of the analogous matter system introduced by Berkovits [2].

The chiral truncation that we are making is possible only if the signature of the four-dimensional spacetime is  $(2, 2)$ . In this case, the  $SO(2, 2)$  Lorentz group is the direct product  $SL(2, R)_L \times SL(2, R)_R$ , with dotted spinorial indices transforming under  $SL(2, R)_L$ , and undotted indices transforming under  $SL(2, R)_R$ . The bosonic currents are singlets under the entire Lorentz group, but the two fermionic currents form a doublet under  $SL(2, R)_L$ .

Unfortunately, these constraints are reducible, which implies that the associated ghosts still have gauge invariances, whose elimination requires the introduction of ghosts for ghosts. In fact the reducibility is of infinite order, just as in the covariant Green-Schwarz superstring, and thus an infinite number of ghosts for ghosts are needed. This problem can be overcome at the price of sacrificing manifest spacetime Lorentz invariance, since in this case we can identify a subset of the constraints that is irreducible, and which also has the critical central charge. Essentially, the

states that are physical under this irreducible subset are also annihilated by the the remaining dependent constraints. (This statement will be made precise for states with standard ghost structure.) In section 2, we discuss the full set of constraints and their irreducible subset. Since the irreducible system is also critical, we obtain the corresponding BRST operator. The irreducible system still maintains  $N = 1$  spacetime supersymmetry. In section 3, we construct some examples of physical states of the BRST operator for the irreducible system, and discuss their interactions. There are two massless physical states which have standard ghost structure, namely a scalar and its spin- $\frac{1}{2}$  superpartner. The physical spectrum also contains an infinite number of massive states.

Although the irreducible system can be solved completely, the spacetime  $SO(2, 2)$  covariance of the original reducible system is broken. It is of interest to have a fully covariant BRST treatment. This system is in one respect slightly simpler than previous examples of reducible systems, such as the covariant Green-Schwarz superstring and the  $N = 4$  string, in that one can build a nilpotent charge  $Q'$  from the reducible constraints. This implies that the standard ghost vacuum of the ghosts for ghosts has zero conformal dimension. Thus if we restrict our attention to physical states that are the tensor product of this vacuum with states in the cohomology of  $Q'$ , the physical-state condition simplifies considerably, and can be discussed without the need to know all the details of the fully covariant BRST operator. We shall discuss the fully covariant BRST procedure for the reducible system in section 4, and discuss the corresponding cohomology in section 5. We shall see that indeed that extra conditions arising from ghosts for ghosts eliminate states of  $Q'$  that have the undesirable feature of carrying infinite-dimensional  $SL(2, R)_L$  representations. The remaining states of the reducible system, after a further truncation of non-interactive states, are expected to coincide with those of the irreducible system. In particular we show that the massless physical states give rise to the identical interactions.

## 2 The constraint algebra and the BRST charge

In this section, we discuss the algebra of constraints that defines the string theory. The matter system consists of the four spacetime coordinates  $X^{\alpha\dot{\alpha}} = \sigma_{\mu}^{\alpha\dot{\alpha}} X^{\mu}$ , the two-component Majorana-Weyl spinor  $\theta^{\alpha}$ , and its conjugate momentum  $p_{\alpha}$ . The action for the matter system takes the form

$$I = \int d^2z \left( -\frac{1}{2} \partial X^{\alpha\dot{\alpha}} \bar{\partial} X_{\alpha\dot{\alpha}} + p_{\alpha} \bar{\partial} \theta^{\alpha} \right). \quad (1)$$

In the language of conformal field theory, these fields satisfy the OPEs

$$X^{\alpha\dot{\alpha}}(z) X^{\beta\dot{\beta}}(w) \sim -\epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} \log(z - w), \quad p_{\alpha}(z) \theta_{\beta}(w) \sim \frac{\epsilon_{\alpha\beta}}{z - w}. \quad (2)$$

When we need to be explicit, we use conventions in which the spacetime metric is given by  $\eta_{\mu\nu} = \text{diag}(-1, -1, 1, 1)$ , the indices  $\mu, \nu \dots$  run from 1 to 4, and the mapping between tensor indices and 2-component spinor indices is defined by

$$V^{\alpha\dot{\alpha}} = \begin{pmatrix} V^{1\dot{1}} & V^{1\dot{2}} \\ V^{2\dot{1}} & V^{2\dot{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} V^1 + V^4 & V^2 - V^3 \\ V^2 + V^3 & -V^1 + V^4 \end{pmatrix}, \quad (3)$$

where  $V^{\mu}$  is an arbitrary vector. The Van der Waerden symbols  $\sigma_{\mu}^{\alpha\dot{\alpha}}$  thus defined satisfy  $\sigma_{\mu}^{\alpha\dot{\alpha}} \sigma^{\mu\beta\dot{\beta}} = \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}}$ , where  $\epsilon_{12} = \epsilon^{12} = 1$ . Spinor indices are raised and lowered according to

the usual “North-west/South-east” convention, with  $\psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta$  and  $\psi_\alpha = \psi^\beta \epsilon_{\beta\alpha}$ , *etc.*, so we have  $\psi^1 = \psi_2$  and  $\psi^2 = -\psi_1$ . Note that since the indices are two-dimensional, we have the useful Schoutens identity  $X^\alpha Y_\alpha Z_\beta + X_\alpha Y_\beta Z^\alpha + X_\beta Y^\alpha Z_\alpha = 0$ .

In [6], Siegel proposed to build a string theory implementing the set of constraints given by  $\{\partial X^{\alpha\dot{\alpha}} \partial X_{\alpha\dot{\alpha}}, p_\alpha \partial \theta^\alpha, p_\alpha p^\alpha, \partial \theta_\alpha \partial \theta^\alpha, p_\alpha \partial X^{\alpha\dot{\alpha}}, \partial \theta_\alpha \partial X^{\alpha\dot{\alpha}}\}$ . However, it follows from (2) that whilst the second order poles in the OPEs amongst this set of constraints give back the same set of constraints, not all the first-order poles can be re-expressed as the derivatives of the constraints. In other words, the algebra does not close. (Note that this non-closure occurs even at the classical level of Poisson brackets, or single OPE contractions.) Accordingly, we choose a subset of Siegel’s constraints that form a closed algebra, namely

$$\begin{aligned} T &= -\frac{1}{2} \partial X^{\alpha\dot{\alpha}} \partial X_{\alpha\dot{\alpha}} - p_\alpha \partial \theta^\alpha, \\ S &= -p_\alpha p^\alpha, \\ G^{\dot{\alpha}} &= -p_\alpha \partial X^{\alpha\dot{\alpha}}. \end{aligned} \tag{4}$$

We see from the energy-momentum tensor that from the worldsheet viewpoint,  $\theta^\alpha$  has conformal weight 0, and  $p_\alpha$  has conformal weight 1. Thus the two bosonic currents  $T$  and  $S$ , and also the two fermionic currents  $G^{\dot{\alpha}}$ , have conformal spin 2. The currents are all primary, and the remaining non-trivial OPE is given by

$$G^{\dot{\alpha}}(z) G^{\dot{\beta}}(w) \sim \frac{2\epsilon^{\dot{\alpha}\dot{\beta}} S}{(z-w)^2} + \frac{\epsilon^{\dot{\alpha}\dot{\beta}} \partial S}{z-w}. \tag{5}$$

The matter currents may be expressed in a concise form by introducing a pair of spin-0 fermionic coordinates  $\zeta^{\dot{\alpha}}$  on the worldsheet. We can then define

$$\mathcal{P}^\alpha = p^\alpha + \zeta_{\dot{\alpha}} \partial X^{\alpha\dot{\alpha}} + \zeta_{\dot{\alpha}} \zeta^{\dot{\alpha}} \partial \theta^\alpha, \tag{6}$$

in terms of which the currents may be written as  $\mathcal{T} = \mathcal{P}_\alpha \mathcal{P}^\alpha$ , where

$$\mathcal{T} = S + \zeta_{\dot{\alpha}} G^{\dot{\alpha}} + \zeta_{\dot{\alpha}} \zeta^{\dot{\alpha}} T. \tag{7}$$

Note that the algebra generated by (4) is a truncation of the “small”  $N = 4$  superconformal algebra that was used to construct the  $N = 4$  string in [7]. This can be seen from the fact that we can augment our currents (4) by including  $\{\theta_\alpha \theta^\alpha, p_\alpha \theta^\alpha, \theta_\alpha \partial X^{\alpha\dot{\alpha}}\}$  as well. One can easily verify that the resulting currents generate precisely the small  $N = 4$  superconformal algebra, in a twisted basis. It was shown in [7] that this realisation of the  $N = 4$  algebra is reducible. In fact the constraints of its  $N = 2$  subalgebra are irreducible, and the associated physical states are also annihilated by the full  $N = 4$  currents. Thus in  $2 + 2$  dimensions, the  $N = 4$  string is equivalent to the  $N = 2$  string [7], which is generally believed not to have spacetime supersymmetry. Our choice of currents (4), which is motivated by the desire to obtain a string theory in  $2 + 2$  dimensions which does have spacetime supersymmetry, generates a different subalgebra of the  $N = 4$  algebra.

Unfortunately, the currents given in (4) are also reducible. Specifically, one can observe that

$$p^\alpha T + \partial X^{\alpha\dot{\alpha}} G_{\dot{\alpha}} + \partial \theta^\alpha S = 0, \quad p^\alpha G^{\dot{\alpha}} + \partial X^{\alpha\dot{\alpha}} S = 0, \quad p_\alpha S = 0. \tag{8}$$

These can be written in the concise form  $\mathcal{P}_\alpha \mathcal{T} = 0$ . As in the case of [7], the reducible constraints (4) can be divided into independent constraints and dependent constraints. The independent constraints can be taken to be

$$T = -\frac{1}{2} \partial X^{\alpha\dot{\alpha}} \partial X_{\alpha\dot{\alpha}} - p_\alpha \partial \theta^\alpha, \quad G^{\dot{1}} = -p_\alpha \partial X^{\alpha\dot{1}}, \quad (9)$$

which in fact generate a subalgebra of the twisted  $N = 2$  superconformal algebra. Using (8), we can write the remaining constraints, *i.e.* the dependent ones, as linear functions of the independent constraints. In momentum space, they are given by

$$S = (p^{\alpha\dot{1}})^{-1} p^\alpha G^{\dot{1}}, \quad G^{\dot{2}} = -(p^{\alpha\dot{1}})^{-1} (p^\alpha T + p^{\alpha\dot{2}} G^{\dot{1}} + \partial \theta^\alpha S), \quad (10)$$

where  $\alpha$  can be chosen to be either 1 or 2. These expressions are valid in the region of phase space where  $p^{\alpha\dot{1}} \neq 0$ . To cover the region where  $p^{\alpha\dot{1}} = 0$ , we can make a different choice of the independent constraints.

The above reducibility of the constraints can be better understood by studying the physical spectrum of the theory. First let us consider the physical operators with standard ghost structure, in which case knowledge of the explicit form of the BRST operators is not necessary. As we shall see later, after imposing the  $T$  and  $G^{\dot{1}}$  constraints, there are two massless physical operators with standard ghost structure, which form a spacetime  $N = 1$  supermultiplet. This pair of operators is then identically annihilated by the remaining constraints  $G^{\dot{2}}$  and  $S$ . This establishes the equivalence between the constraints of (4) and the reduced ones (9), for the massless physical states. However there are further massive operators with standard ghost structure under only the  $T$  and  $G^{\dot{1}}$  constraints, which do not seem to be annihilated by the constraints of  $G^{\dot{2}}$  and  $S$ . To establish the equivalence of the massive spectra of the reducible and the irreducible systems would require the analysis of the full cohomology and interactions, including the physical states with non-standard ghost structure.

In order to discuss the physical spectrum with non-standard ghost structure, it is necessary to obtain the explicit form of the BRST operators. The construction of the BRST operator for a system with reducible constraints is discussed in [8]. The reducibility implies that the ghosts for the original constraints still have gauge invariances, whose elimination requires the introduction of ghosts for ghosts. As in the case of covariant quantisation of the Green-Schwarz string, the reducibility relations (8) are themselves overcomplete; in fact the system has an infinite order of reducibility. This can be easily seen from the form  $\mathcal{P}_\alpha \mathcal{T} = 0$  for the reducibility relations, owing to the fact that the functions  $\mathcal{P}_\alpha$  are themselves reducible, since  $\mathcal{P}_\alpha \mathcal{P}^\alpha$  gives back the constraints  $\mathcal{T}$ . This infinite order of reducibility implies that a proper BRST treatment requires an infinite number of ghosts for ghosts. The form of the BRST operator, after making the decomposition into independent constraints  $(T, G^{\dot{1}})$  and dependent constraints  $(G^{\dot{2}}, S)$ , is

$$\tilde{Q} = Q + \sum_{k \geq 0} \hat{b}_{\alpha_k} c^{\alpha_{k+1}}, \quad (11)$$

where  $Q$  is the standard BRST operator for the irreducible system described by  $(T, G^{\dot{1}})$ ,  $\hat{b}_{\alpha_k}$  are the level- $k$  antighosts for the dependent constraints, and  $c^{\alpha_k}$  are the level- $k$  ghosts for the independent constraints. Since the ghost fields in the second term do not appear in  $Q$ , and they form Kugo-Ojima quartets, the BRST cohomology of  $\tilde{Q}$  is equivalent to that of  $Q$ . To see this, note that any states with excitations of  $\hat{c}^{\alpha_k}$  or  $b_{\alpha_k}$  will not be annihilated by the BRST operator,

whilst any states with excitations of  $\hat{b}_{\alpha_k}$  or  $c^{\alpha_k}$  are BRST trivial, since  $\{\tilde{Q}, b_{\alpha_{k+1}}\} = \hat{b}_{\alpha_k}$  and  $\{\tilde{Q}, \hat{c}^{\alpha_{k-1}}\} = -c^{\alpha_k}$ . Although the BRST operator  $\tilde{Q}$  was obtained at the classical level, it is also nilpotent at the quantum level.

The price that we have paid for the simple form of the BRST operator  $\tilde{Q}$  is that the  $SL(2, R)_L$  covariance of the  $\dot{\alpha}$  indices in the original reducible constraints has been sacrificed. The fully covariant BRST treatment of the reducible system remains to be understood. We shall now proceed by constructing the BRST operator  $Q$  for the irreducible system  $(T, G^{\dot{1}})$ . We begin by introducing the anticommuting ghosts  $(b, c)$  and the commuting ghosts  $(r, s)$  for  $T$  and  $G^{\dot{1}}$  respectively. The commuting ghosts  $(r, s)$  are bosonised, *i.e.*  $r = \partial\xi e^{-\phi}$ ,  $s = \eta e^{\phi}$ . In terms of these fields, the BRST operator  $Q$  is given by

$$Q = c \left( -\frac{1}{2} \partial X^{\alpha\dot{\alpha}} \partial X_{\alpha\dot{\alpha}} - p_{\alpha} \partial \theta^{\alpha} - b \partial c - \frac{1}{2} (\partial \phi)^2 - \frac{3}{2} \partial^2 \phi - \eta \partial \xi \right) + \eta e^{\phi} p_{\alpha} \partial X^{\alpha\dot{1}}. \quad (12)$$

The theory has spacetime supersymmetry, generated by

$$q^{\alpha} = \oint p^{\alpha}, \quad q^{\dot{1}} = \oint \theta_{\alpha} \partial X^{\alpha\dot{1}}, \quad q^{\dot{2}} = \oint \theta_{\alpha} \partial X^{\alpha\dot{2}} + b \eta e^{\phi}. \quad (13)$$

The somewhat unusual ghost terms in  $q^{\dot{2}}$  are necessary for the generator to anti-commute with the BRST operator. It is straightforward to verify that these supercharges generate the usual  $N = 1$  spacetime superalgebra

$$\{q_{\alpha}, q_{\beta}\} = 0 = \{q^{\dot{\alpha}}, q^{\dot{\beta}}\}, \quad \{q^{\alpha}, q^{\dot{\alpha}}\} = P^{\alpha\dot{\alpha}}, \quad (14)$$

where  $P^{\alpha\dot{\alpha}} = \oint \partial X^{\alpha\dot{\alpha}}$  is the spacetime translation operator.

Since the zero mode of  $\xi$  is not included in the Hilbert space of physical states, there exists a BRST non-trivial picture-changing operator  $Z = \{Q, \xi\}$  which can give new BRST non-trivial physical operators when normal ordered with others. Explicitly, it takes the form

$$Z = c \partial \xi + p_{\alpha} \partial X^{\alpha\dot{1}} e^{\phi}. \quad (15)$$

Unlike the picture-changing operator in the usual  $N = 1$  NSR superstring, this operator has no inverse.

### 3 Physical states and interactions

In this section, we shall discuss the cohomology of the BRST operator  $Q$  given in (12) for the irreducible system  $(T, G^{\dot{1}})$ , and present some results for the physical states in the theory. We begin by studying the physical spectrum with standard ghost structure. There are two massless operators

$$V = c e^{-\phi} e^{ip \cdot X}, \quad \Psi = h_{\alpha} c e^{-\phi} \theta^{\alpha} e^{ip \cdot X}, \quad (16)$$

which are physical provided with mass-shell condition  $p^{\alpha\dot{\alpha}} p_{\alpha\dot{\alpha}} = 0$  and spinor polarisation condition  $p^{\alpha\dot{1}} h_{\alpha} = 0$ . The non-triviality of these operators can be established by the fact that the conjugates of these operators with respect to the following non-vanishing inner product

$$\left\langle \partial^2 c \partial c c e^{-3\phi} \theta^2 \right\rangle \quad (17)$$

are also annihilated by the BRST operator. The bosonic operator  $V$  and the fermionic operator  $\Psi$  form a supermultiplet under the  $N = 1$  spacetime supersymmetric transformation. The associated spacetime fields  $\phi$  and  $\psi_\alpha$  transform as

$$\delta\phi = \epsilon_\alpha \psi^\alpha \quad \delta\psi_\alpha = \epsilon^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \phi . \quad (18)$$

We can build only one three-point amplitude among the massless operators, namely

$$\langle V(z_1) \Psi(z_2) \Psi(z_3) \rangle = c_{23} , \quad (19)$$

where  $b_{ij}$  is defined by

$$b_{ij} = h_{(i)\alpha} h_{(j)}^\alpha . \quad (20)$$

From this, we can deduce that the  $V$  operator describes a spacetime scalar whilst the  $\Psi$  operator describes a spacetime chiral spin- $\frac{1}{2}$  fermion. Note that this is quite different from the case of the  $N = 2$  string where there is only a massless boson and although it is ostensibly a scalar, it in fact, as emerges from the study of the three-point amplitudes, a prepotential for self-dual Yang-Mills or gravity. With the one insertion of the picture-changing operator, we can build a four-point function which vanishes for kinematic reasons:

$$\langle ZV \Psi \oint b\Psi \Psi \rangle = (u b_{12} b_{34} + s b_{13} b_{24}) \frac{\Gamma(-\frac{1}{2}s) \Gamma(-\frac{1}{2}t)}{\Gamma(\frac{1}{2}u)} , \quad (21)$$

where  $s$ ,  $t$ , and  $u$  are the Mandelstam variables and  $h_{(1)}^\alpha = p_{(1)}^{\alpha 1}$ . The vanishing of the kinematic term, *i.e.*

$$u b_{12} b_{34} + s b_{13} b_{24} = 0 \quad (22)$$

is a straightforward consequence of the mass-shell condition of the operators and momentum conservation of the four-point amplitude [4]. It might seem that the vanishing of the this four-point amplitude should be automatically implied by the statistics of the operators since there is an odd number of fermions. However, as we shall see later, the picture-changing operator has spacetime fermionic statistics. In fact, that the four-point amplitude (21) vanishes only on-shell, for kinematic reasons (22), already implies that the picture changer  $Z$  is a fermion. Thus the picture changing of a physical operator changes its spacetime statistics and hence does not establish the equivalence between the two. On the other hand, since  $Z^2 = (ZZ)$  becomes a spacetime bosonic operator, we can use  $Z^2$  to identify the physical states with different pictures. Thus we have a total of four massless operators, namely  $V$ ,  $ZV$  and their supersymmetric partners.  $V$  and its superpartner  $\Psi$  have standard ghost structure;  $ZV$  and its superpartner  $Z\Psi$  have non-standard ghost structures.

So far we have constructed massless physical states. There are also infinitely many massive states. The tachyonic type of massive operators, *i.e.* those that become pure exponentials after bosonising the fermionic fields, can be easily obtained. They are given by

$$\begin{aligned} V_n &= c(\partial^n p)^2 \dots p^2 e^{n\phi} e^{ip \cdot X} , & \mathcal{M}^2 &= (n+1)(n+2) , \\ \tilde{V}_n &= c(\partial^{n+1} \theta)^2 \dots \theta^2 e^{-(n+3)\phi} e^{ip \cdot X} , & \mathcal{M}^2 &= (n+1)(n+2) , \\ U_n^{(\delta)} &= h_\alpha^{(\delta)} c \partial^n p^\alpha (\partial^{n-1} p)^2 \dots p^2 e^{(n-1+\delta)\phi} e^{ip \cdot X} , & \mathcal{M}^2 &= (n+1)(n+2-2\delta) , \\ \tilde{U}_n^{(\delta)} &= \tilde{h}_\alpha^{(\delta)} c \partial^{n+1} \theta^\alpha (\partial^n \theta)^2 \dots \theta^2 e^{-(n+2+\delta)\phi} e^{ip \cdot X} , & \mathcal{M}^2 &= (n+1)(n+2-2\delta) , \\ W_n &= h_{\alpha\beta} c \partial^{n+1} p^\alpha \partial^n p^\beta (\partial^{n-1} p)^2 \dots p^2 e^{n\phi} e^{ip \cdot X} , & \mathcal{M}^2 &= (n^2 + 3n + 4) , \\ \tilde{W}_n &= \tilde{h}_{\alpha\beta} c \partial^{n+2} \theta^\alpha \partial^{n+1} \theta^\beta (\partial^n \theta)^2 \dots \theta^2 e^{-(n+3)\phi} e^{ip \cdot X} , & \mathcal{M}^2 &= (n^2 + 3n + 4) , \end{aligned} \quad (23)$$

where  $p^2 = p_\alpha p^\alpha$ , *etc.* and  $\delta = 0$  or  $1$ .  $V_n$  and  $\tilde{V}_n$  are physical provided the proper mass-shell condition is satisfied. For the remaining operators, in addition to mass-shell conditions, they also satisfy certain spinor polarisation conditions:  $p^{\alpha\dot{1}} h_\alpha^{(1)} = 0$ ,  $p^{\alpha\dot{1}} \tilde{h}_\alpha^{(0)} = 0$ ,  $p^{\alpha\dot{2}} h_\alpha^{(0)} = 0$ ,  $p^{\alpha\dot{2}} \tilde{h}_\alpha^{(1)} = 0$ ,  $p^{\alpha\dot{1}} h_{\alpha\beta} = 0$  and  $p^{\alpha\dot{1}} \tilde{h}_{\alpha\beta} = 0$ . In addition  $h_{\alpha\beta}$  and  $\tilde{h}_{\alpha\beta}$  must be symmetric.

The spacetime statistics of the physical operators can be determined as follows. The non-vanishing of the three-point amplitudes  $\langle \tilde{V}_{2n} U_n^{(1)} U_n^{(1)} \rangle$  and  $\langle V_{2n+1} \tilde{U}_n^{(0)} \tilde{U}_n^{(0)} \rangle$  implies that  $V_{2n+1}$  and  $\tilde{V}_{2n}$  are bosons. Since  $\partial c \tilde{V}_{2n+1}$  and  $\partial c V_{2n}$  are their conjugates, it follows that all  $V_n$  and  $\tilde{V}_n$  are bosons. On the other hand, the non-vanishing of the three-point amplitudes  $\langle \Psi U_n^{(1)} \tilde{V}_{n-1} \rangle$  and  $\langle \Psi \tilde{U}_n^{(0)} V_n \rangle$  leads to the conclusion that  $U_n^{(\delta)}$  and  $\tilde{U}_n^{(\delta)}$  are fermions. There are various relations among the above physical operators. For example,  $ZU_n^{(0)} = V_n$  and  $ZV_n = U_{n+1}^{(1)}$ . Thus we can immediately see that the picture changer  $Z$  is a spacetime fermion, and hence the above relations do not establish the equivalence between the  $U$  and  $V$  operators. However, they do imply that  $U_n^{(0)}$  and  $U_{n+1}^1$  are equivalent. The superpartners of these operators can be obtained by the action of the supersymmetry generators given in (13).

The above operators are only a small subset of the complete spectrum of massive operators. The  $(\text{mass})^2$  of these operators grows quadratically with  $n$ . However, as we shall show, one can build non-vanishing four-point amplitudes from these operators, which implies the existence of massive operators with the usual linear growth. The simplest four-point amplitude that can be built is given by

$$\langle U_3^{(1)} \tilde{U}_0^{(0)} \oint b \tilde{U}_0^{(0)} \tilde{U}_0^{(0)} \rangle = \left( A(\tfrac{1}{2}u - 3) + B(\tfrac{1}{2}s - 3) \right) \frac{\Gamma(-\tfrac{1}{2}s + 3)\Gamma(-\tfrac{1}{2}t + 3)}{\Gamma(\tfrac{1}{2}u - 2)}, \quad (24)$$

where  $s, t, u$  are the Mandelstam variables, and  $A, B$  are given by

$$A = h_{(1)}^\alpha h_{(2)\alpha} h_{(3)\beta} h_{(4)}^\beta, \quad B = h_{(1)}^\alpha h_{(2)\beta} h_{(3)\alpha} h_{(4)}^\beta. \quad (25)$$

The physical-state conditions imply that  $p_{(1)\alpha\dot{\alpha}}^{\alpha\dot{\alpha}} = -12$ ,  $p_{(i)\alpha\dot{\alpha}}^{\alpha\dot{\alpha}} = -2$ , for  $i = 2, 3, 4$ , and  $p_{(i)}^{\alpha\dot{1}} h_{(i)\alpha} = 0$  for  $i = 1, 2, 3, 4$ . Thus the solution for these polarisation spinors is  $h_{(i)}^\alpha = p_{(i)}^{\alpha\dot{1}}$  (up to arbitrary scaling factors). One can then easily verify that the prefactor of the four-point amplitude (24) does not vanish. This implies that there are further massive operators with  $(\text{mass})^2 = 2(n+3)$  in the spectrum. Moreover this result is also consistent with the fact that the picture-changing operator  $Z$  is a spacetime fermion. To see this, we first note that the four-point amplitude (24) can be restated as  $\langle ZV_2 \tilde{U}_0^{(0)} \tilde{U}_0^{(0)} \tilde{U}_0^{(0)} \rangle$ . Since  $V_2$  is spacetime boson whilst  $\tilde{U}_0^{(0)}$  is spacetime fermion, it follows that the non-vanishing of the four-point amplitude implies that the picture changer  $Z$  is a spacetime fermion.

The massive physical operators that we found explicitly in (23) all have non-standard ghost structures. From these operators, we can build non-vanishing four-point amplitudes, which implies the existence of further massive operators in the physical spectrum. The structure of these massive operators that are exchanged in the four-point amplitudes can be determined by the structures of the external physical operators. In particular, the non-vanishing four-point amplitude  $\langle U_n^{(1)} \tilde{U}_{n-1}^{(0)} \Psi \Psi \rangle$  implies, by looking at the  $s$  channel, the existence of massive operators with standard ghost structure.<sup>3</sup>

<sup>3</sup>We thank the referee for drawing our attention to the existence of massive operators with standard ghost structure.



## 4 Covariant quantisation of the reducible system

In the previous section, we discussed the physical spectrum and interactions for the irreducible system  $(T, G^1)$ . Although the theory is spacetime supersymmetric, the manifest  $SO(2, 2) \sim SL(2, R)_L \times SL(2, R)_R$  covariance of the original reducible system is partially broken down to  $SL(2, R)_R$ . The fully covariant BRST treatment of the original reducible system is far more complicated, since the ghost for ghost terms no longer decouple. This is a consequence of the fact that the reducibility relations  $\mathcal{P}^\alpha \mathcal{T} = 0$  have an infinite order of reducibility since  $\mathcal{P}^\alpha \mathcal{P}_\alpha = \mathcal{T}$ , which is reminiscent of the situation for the covariant quantisation of the Green-Schwarz superstring. Another example is the string theory in  $2 + 2$  dimensions associated with the small  $N = 4$  superconformal algebra, which was discussed by Siegel in [7]. Naively the central charge for the  $N = 4$  string is  $-12$ ; however, in  $2 + 2$  dimensions the constraints are reducible. The irreducible subset of the constraints generates the  $N = 2$  superconformal algebra, for which the critical dimension is indeed four. Thus the proper fully covariant BRST treatment of the reducible  $N = 4$  string requires the introduction of ghosts for ghosts that will contribute to the central charge for criticality. In one respect our example is slightly simpler, in that the matter has the critical central charge both for the irreducible system and the original reducible system. This means that we can write down a nilpotent charge  $Q'$  for the reducible system without the need of ghost for ghost terms. It is of course not the true BRST charge for the corresponding string theory. However since  $Q'$  is already nilpotent, the contributions to the central charge from the ghosts for ghosts should be zero. We shall see later that this feature makes the analysis of the cohomology much simpler.

The fully covariant BRST procedure for a reducible system is described in [8]. As we showed in section 2, the matter currents can be expressed in a concise form (7) and (6) by introducing a pair of spin-0 fermionic coordinates  $\zeta^\alpha$  on the worldsheet. Analogously, the ghosts and antighosts can also be written in the form:

$$\begin{aligned} \mathcal{C}_k &= c_k + \zeta_{\dot{\alpha}} s_k^{\dot{\alpha}} + \zeta_{\dot{\alpha}} \zeta^{\dot{\alpha}} \gamma_k , \\ \mathcal{B}_k &= \beta_k + \zeta_{\dot{\alpha}} r_k^{\dot{\alpha}} + \zeta_{\dot{\alpha}} \zeta^{\dot{\alpha}} b_k , \end{aligned} \quad (26)$$

where the index  $k$  denotes the level of the ghosts for ghosts. In terms of these fields, the fully covariant BRST operator can be written as

$$Q = \mathcal{C}_0 \left( \mathcal{T} + \partial \mathcal{C}_0 \mathcal{B}_0 + \sum_{k \geq 1} ((k+2) \partial \mathcal{C}_k \mathcal{B}_k - (k+1) \mathcal{C}_k \partial \mathcal{B}_k) \right) + \sum_{k \geq 0} \mathcal{C}_{k+1} \mathcal{P} \mathcal{B}_k + \text{“more”} . \quad (27)$$

The level- $k$  ghosts and antighosts carry an  $\alpha$  index when  $k$  is odd and no index when  $k$  is even. We have suppressed these  $\alpha$  indices, and the  $\alpha$  index on  $\mathcal{P}_\alpha$ , in the above expression. The “more” term involves pure-ghost expressions that are needed for the nilpotency of the BRST operator. Since the first two terms in the above expression, *i.e.* the “BRST” operator for the original reducible system,

$$Q' = \mathcal{C}_0 \mathcal{T} - \partial \mathcal{C}_0 \mathcal{C}_0 \mathcal{B}_0 , \quad (28)$$

is already a nilpotent operator, it implies that the contributions from higher level ghosts for ghosts are zero. Indeed, the contributions are zero level by level, owing to a cancellation between the contributions from the commuting and anticommuting ghosts for ghosts.

The complete expression for the BRST operator (27) is very complicated, and is not yet known. Thus a full analysis of its cohomology is not possible at present. However, owing to the

feature we discussed above, we can restrict our attention to the physical states with standard ghost structure in the  $k \geq 1$  ghost sectors, *i.e.* the physical states of the form

$$|\text{phys}\rangle = V|0\rangle = V'|\text{gh}\rangle_{k \geq 1} , \quad (29)$$

where  $|\text{gh}\rangle_{k \geq 1}$  is the standard ghost vacuum for the ghosts for ghosts, and  $V'$  is an operator which involves only the matter and  $k = 0$  ghosts. For states of this form, the physical-state condition  $Q|\text{phys}\rangle = 0$  involves the following relevant terms in the BRST operator (27):

$$Q \sim Q' + \sum_{k \geq 0} \mathcal{C}_{k+1} \mathcal{P} \mathcal{B}_k . \quad (30)$$

Thus for physical states, we must have

$$Q' V'|0\rangle = 0 , \quad (31)$$

$$\left( \sum_{k \geq 0} \mathcal{C}_{k+1} \mathcal{P} \mathcal{B}_k \right) V'|\text{gh}\rangle_{k \geq 1} = 0 . \quad (32)$$

These states will be non-trivial if the operator  $V'$  is non-trivial with respect to  $Q'$ . In the following section, we shall therefore study the cohomology of the nilpotent charge  $Q'$ . Then we shall discuss the extra condition (32) on these states to obtain the physical states of the fully covariant BRST operator (27).

## 5 Cohomology of the reducible system

In the previous section, we discussed the fully covariant BRST quantisation of the reducible system (4). It requires the introduction of infinitely many ghosts for ghosts. In this section, we shall study the cohomology of the BRST operator with the physical states that are of the form (29). As we discussed in the previous section, for the physical states of this form it is convenient first to discuss the cohomology of the nilpotent charge  $Q'$ , and then we shall examine the extra conditions (32) arising from the introduction of the ghosts for ghosts

For simplicity, we shall discuss the cohomology of the nilpotent BRST operator  $Q'$  in (28) in component language. The matter current  $\mathcal{T}$  is defined in (7) with components defined in (4). The ghosts and antighosts  $\mathcal{C}_0, \mathcal{B}_0$  are defined in (26). We shall from now on suppress the index 0 and we shall also refer to the nilpotent operator  $Q'$  as a “BRST” operator. We introduce anticommuting ghosts  $(b, c)$  and  $(\beta, \gamma)$  for the bosonic currents  $T$  and  $S$ , and commuting ghosts  $(r^{\dot{\alpha}}, s_{\dot{\alpha}})$  for the fermionic currents  $G^{\dot{\alpha}}$ , with  $r^{\dot{\alpha}}(z)s^{\dot{\beta}}(w) \sim -\epsilon^{\dot{\alpha}\dot{\beta}}(z-w)^{-1}$ . All anti-ghosts  $(b, \beta, r^{\dot{\alpha}})$  have spin 2, and all ghosts  $(c, \gamma, s_{\dot{\alpha}})$  have spin  $-1$ . Straightforward computation leads to the following result for the nilpotent operator  $Q'$  (28) in terms of the component language:

$$Q' = Q_0 + Q_1 + Q_2 , \quad (33)$$

$$Q_0 = \oint c \left( -\frac{1}{2} \partial X^{\alpha\dot{\alpha}} \partial X_{\alpha\dot{\alpha}} - p_{\alpha} \partial \theta^{\alpha} - b \partial c - 2\beta \partial \gamma - \partial \beta \gamma + 2r^{\dot{\alpha}} \partial s_{\dot{\alpha}} + \partial r^{\dot{\alpha}} s_{\dot{\alpha}} \right) , \quad (34)$$

$$Q_1 = \frac{1}{2} \oint \gamma p_{\alpha} p^{\alpha} , \quad (35)$$

$$Q_2 = \oint (s_{\dot{\alpha}} G^{\dot{\alpha}} - \beta s_{\dot{\alpha}} \partial s^{\dot{\alpha}}) . \quad (36)$$

Under  $SL(2, R)_R$  transformations, corresponding to the self-dual Lorentz transformations of the undotted indices, the currents are all invariant. Thus the generators of  $SL(2, R)_R$  are simply given by

$$J^{\alpha\beta} = \oint \left( -\frac{1}{2} X^{(\alpha} \partial X^{\beta)\dot{\alpha}} + p^{(\alpha} \theta^{\beta)} \right), \quad (37)$$

and, for an infinitesimal transformation with (symmetric) parameter  $\omega_{\alpha\beta}$ , have the action  $\delta\psi^\alpha = [\omega_{\beta\gamma} J^{\beta\gamma}, \psi^\alpha] = \omega^\alpha{}_\beta \psi^\beta$  on any undotted index. The  $SL(2, R)_L$  transformations, on the other hand, which correspond to anti-self-dual Lorentz rotations of the dotted indices, rotate the fermionic currents  $G^{\dot{\alpha}}$ , and hence the ghosts  $(r^{\dot{\alpha}}, s_{\dot{\alpha}})$  must rotate also. It is quite easy to see that the generators are given by

$$J^{\dot{\alpha}\dot{\beta}} = \oint \left( \frac{1}{2} X^{\alpha(\dot{\alpha}} \partial X_{\alpha}^{\dot{\beta})} + r^{(\dot{\alpha}} s^{\dot{\beta})} \right), \quad (38)$$

and they transform dotted indices according to  $\delta\psi^{\dot{\alpha}} = \omega^{\dot{\alpha}}{}_{\dot{\beta}} \psi^{\dot{\beta}}$ . These spacetime Lorentz transformations are symmetries of the two-dimensional action including ghosts, and they commute with the BRST charge.

It is also useful to write down the form of the generators of manifest spacetime supersymmetry. They are given by

$$q_\alpha = \oint p_\alpha, \quad (39)$$

$$q^{\dot{\alpha}} = \oint \left( -\theta_\alpha \partial X^{\alpha\dot{\alpha}} - \gamma r^{\dot{\alpha}} + b s^{\dot{\alpha}} \right). \quad (40)$$

The somewhat unusual ghost terms in  $q^{\dot{\alpha}}$  are a consequence of the fact that  $r^{\dot{\alpha}}$  and  $s_{\dot{\alpha}}$  transform under the anti-self-dual spacetime Lorentz group. It is straightforward to verify that these supercharges generate the usual  $N = 1$  spacetime superalgebra (14).

As usual in a theory with fermionic currents, it is appropriate to bosonise the associated commuting ghosts. Thus we write

$$r^{\dot{\alpha}} = \partial \xi^{\dot{\alpha}} e^{-\phi_{\dot{\alpha}}}, \quad s_{\dot{\alpha}} = \eta_{\dot{\alpha}} e^{\phi_{\dot{\alpha}}}, \quad (41)$$

where  $\eta_{\dot{\alpha}}$  and  $\xi^{\dot{\alpha}}$  are anticommuting fields with spins 1 and 0 respectively. The OPEs of the bosonising fields are  $\eta_{\dot{\alpha}}(z) \xi^{\dot{\beta}}(w) \sim \delta_{\dot{\alpha}}^{\dot{\beta}} (z-w)^{-1}$ , and  $\phi_{\dot{\alpha}}(z) \phi_{\dot{\beta}}(w) \sim -\delta_{\dot{\alpha}\dot{\beta}} \log(z-w)$ . Note that the bosonisation breaks the manifest  $SL(2, R)_L$  covariance, and that the  $\dot{\alpha}$  index in (41) is not summed. In view of this non-covariance, there is no particular advantage in using upper as well as lower indices on  $\phi_{\dot{\alpha}}$ , and we find it more convenient always to use lower ones for this purpose.

The nilpotent operator  $Q'$  can be easily re-expressed in terms of the bosonised fields; the  $(r, s)$  terms in  $Q_0$  become  $\oint c \left( -\eta_{\dot{\alpha}} \partial \xi^{\dot{\alpha}} - \frac{1}{2} (\partial \phi_1)^2 - \frac{1}{2} (\partial \phi_2)^2 - \frac{3}{2} \partial^2 \phi_1 - \frac{3}{2} \partial^2 \phi_2 \right)$ , whilst  $Q_2$  becomes

$$\begin{aligned} Q_2 &= \oint \left( \eta_1 p_\alpha \partial X^{\alpha 1} e^{\phi_1} + \eta_2 p_\alpha \partial X^{\alpha 2} e^{\phi_2} \right) \\ &+ \oint \beta \left( \eta_1 \partial \eta_2 - \partial \eta_1 \eta_2 - \eta_1 \eta_2 (\partial \phi_1 - \partial \phi_2) \right) e^{\phi_1 + \phi_2}. \end{aligned} \quad (42)$$

(It is to be understood that an expression such as  $e^{\phi_1 + \phi_2}$  really means :  $e^{\phi_1} :: e^{\phi_2}$  :, which equals - :  $e^{\phi_2} :: e^{\phi_1}$  : since both of these exponentials are fermions. Thus we have  $e^{\phi_1 + \phi_2} = -e^{\phi_2 + \phi_1}$  in this rather elliptical notation.)

The ghost contributions to the  $SL(2, R)_L$  Lorentz generators (38) become

$$\begin{aligned} J_+ &= r_1 s_1 = \eta_1 \partial \xi^2 e^{\phi_1 - \phi_2} , \\ J_- &= r_2 s_2 = \eta_2 \partial \xi^1 e^{-\phi_1 + \phi_2} , \end{aligned} \quad (43)$$

$$J_3 = r_{(1} s_2) = -\frac{1}{2}(\partial \phi_1 - \partial \phi_2) . \quad (44)$$

One can easily see that these generate an  $SL(2, R)$  Kac-Moody algebra. The translation of the supersymmetry charges into bosonised form is obtained by simple substitution.

Since the zero modes of the  $\xi^{\dot{\alpha}}$  fields are not included in the Hilbert space of physical states, there exist BRST non-trivial picture-changing operators  $Z^{\dot{\alpha}} = \{Q', \xi^{\dot{\alpha}}\}$  which can give new BRST non-trivial physical operators when normal ordered with others. Explicitly, they take the form

$$Z^1 = c \partial \xi^1 - p_{\alpha} \partial X^{\alpha 1} e^{\phi_1} - (2\beta \partial \eta_2 + \partial \beta \eta_2 + 2\beta \eta_2 \partial \phi_2) e^{\phi_1 + \phi_2} , \quad (45)$$

$$Z^2 = c \partial \xi^2 - p_{\alpha} \partial X^{\alpha 2} e^{\phi_2} - (2\beta \partial \eta_1 + \partial \beta \eta_1 + 2\beta \eta_1 \partial \phi_1) e^{\phi_1 + \phi_2} , \quad (46)$$

Like the case of the irreducible system, these operators have no inverse.

## 5.1 Physical states of $Q'$

### 5.1.1 Preliminaries

In this subsection, we shall discuss the cohomology of the nilpotent operator  $Q'$ . Owing to the rather unusual feature in this theory that some of the ghosts carry target spacetime spinor indices, the notion of the standard ghost vacuum requires some modification. We begin by noting that the non-vanishing correlation function that defines the meaning of conjugation is given by

$$\langle \partial^2 c \partial c c \partial^2 \gamma \partial \gamma \gamma e^{-3\phi_1 - 3\phi_2} \theta^2 \rangle \neq 0 , \quad (47)$$

where  $\theta^2 \equiv \theta^{\alpha} \theta_{\alpha}$ . In terms of the bosonised form of the commuting ghosts, the usual operator  $e^{-\phi_1 - \phi_2}$  appearing in the definition of the ghost vacuum can be generalised to an operator  $W_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$ , totally symmetric in its indices, whose component with  $(s+m)$  indices taking the value  $\dot{1}$  and  $(s-m)$  taking the value  $\dot{2}$  is given by

$$W_{\dot{1} \dots \dot{1} \dot{2} \dots \dot{2}} = \lambda(s, m) \partial^{s+m-1} \eta_1 \dots \partial \eta_1 \eta_1 \partial^{s-m-1} \eta_2 \dots \partial \eta_2 \eta_2 e^{(s+m-1)\phi_1 + (s-m-1)\phi_2} . \quad (48)$$

The normalisation constants  $\lambda(s, m)$  are given by

$$\lambda(s, m) = \prod_{p=1}^{s+m-1} \prod_{q=1}^{s-m-1} \frac{1}{p! q!} , \quad (49)$$

where any product over an empty range is defined to be 1.  $W$  in (48) has  $(s+m)$  factors involving  $\eta_1$ , and  $(s-m)$  factors involving  $\eta_2$ , with  $-s \leq m \leq s$ . It is the  $J_3 = m$  state in the  $(2s+1)$ -dimensional spin- $s$  representation of  $SL(2, R)_L$ . The operator  $W_{\dot{1} \dots \dot{1}}$  corresponds to the highest-weight state in the representation, satisfying  $J_+ W_{\dot{1} \dots \dot{1}} = 0$ , with the remaining  $2s$  states being obtained by acting repeatedly with  $J_-$ , each application of which turns a further “ $\dot{1}$ ” index into a “ $\dot{2}$ ”, until the lowest-weight state  $W_{\dot{2} \dots \dot{2}}$  is obtained. Note, incidentally, that the form of the states given in (48) becomes rather simple if one bosonises the  $(\eta, \xi)$  fields.

We may also define a “conjugate” operator  $\widetilde{W}^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$ , again totally symmetric in its indices, by

$$\widetilde{W}^{\dot{1} \dots \dot{1} \dot{2} \dots \dot{2}} = \tilde{\lambda}(s, m) \partial^{s+m} \xi^1 \dots \partial^2 \xi^1 \partial \xi^1 \partial^{s-m} \xi^2 \dots \partial^2 \xi^2 \partial \xi^2 e^{-(s+m+2)\phi_1 - (s-m+2)\phi_2} , \quad (50)$$

with

$$\tilde{\lambda}(s, m) = \prod_{p=1}^{s+m} \prod_{q=1}^{s-m} \frac{1}{p! q!} . \quad (51)$$

Thus the usual ghost vacuum operator  $e^{-\phi_1 - \phi_2}$  and its “conjugate”  $e^{-2\phi_1 - 2\phi_2}$  correspond to the  $s = 0$  cases  $W$  and  $\widetilde{W}$  respectively. All the operators  $W_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$  and  $\widetilde{W}^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$  have worldsheet conformal spin 2, and they all have the property of defining vacuum states that are annihilated by the positive Laurent modes of  $r^{\dot{\alpha}}$  and  $s_{\dot{\alpha}}$ , but not by the negative modes.

These operators have simple properties when acted on by  $Q'$ . The relevant facts can be summarised in the following lemmas:

$$Q_2 W_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} \theta^\alpha e^{ip \cdot X} = ip^{\alpha \dot{\alpha}_{2s+1}} W_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s+1}} e^{ip \cdot X} , \quad (52)$$

$$Q_2 \widetilde{W}^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} \theta^\alpha e^{ip \cdot X} = ip^{\alpha(\dot{\alpha}_1} \widetilde{W}^{\dot{\alpha}_2 \dots \dot{\alpha}_{2s})} e^{ip \cdot X} . \quad (53)$$

A factor of  $\gamma$  or  $\partial\gamma\gamma$  may be included on both sides of the equation in either formula. Note that in (53), the right-hand side is defined to be zero if  $s = 0$ . It is worth remarking that we have recovered the manifest covariance under  $SL(2, R)_L$  in the expressions for the  $W^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$  and  $\widetilde{W}^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$ , even though it was broken by the bosonisation of the ghosts.

### 5.1.2 Physical states

Let us first consider massless operators in the physical spectrum. There are four types of massless operators that can be built from  $W$  operators, namely

$$\begin{aligned} U &= h_{\alpha \dot{\alpha}_1 \dots \dot{\alpha}_{2s}} c \gamma W^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} \theta^\alpha e^{ip \cdot X} , \\ V &= g_{\alpha \dot{\alpha}_1 \dots \dot{\alpha}_{2s}} c \partial\gamma\gamma W^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} \theta^\alpha e^{ip \cdot X} , \\ \Psi &= h_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} c \gamma W^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} e^{ip \cdot X} , \\ \Phi &= g_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} c \partial\gamma\gamma W^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} e^{ip \cdot X} , \end{aligned} \quad (54)$$

and there are four types of physical operators that are associated with the operator  $\widetilde{W}$

$$\begin{aligned} \widetilde{U} &= \tilde{h}_{\alpha \dot{\alpha}_1 \dots \dot{\alpha}_{2s}} c \gamma \widetilde{W}^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} \theta^\alpha e^{ip \cdot X} , \\ \widetilde{V} &= \tilde{g}_{\alpha \dot{\alpha}_1 \dots \dot{\alpha}_{2s}} c \partial\gamma\gamma \widetilde{W}^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} \theta^\alpha e^{ip \cdot X} , \\ \widetilde{\Psi} &= \tilde{h}_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} c \gamma \widetilde{W}^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} \theta^2 e^{ip \cdot X} , \\ \widetilde{\Phi} &= \tilde{g}_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} c \partial\gamma\gamma \widetilde{W}^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} \theta^2 e^{ip \cdot X} . \end{aligned} \quad (55)$$

These operators are the conjugates of the physical operators (54) with  $\partial c c \longrightarrow c$ . We shall thus only discuss the physical-state condition of the states in (54)

We find that  $U$  itself is annihilated by  $Q'$  provided that the following conditions hold:

$$p^{\alpha \dot{\alpha}} p_{\alpha \dot{\alpha}} = 0 , \quad h^{\alpha(\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} p_{\alpha}^{\dot{\alpha})} = 0 . \quad (56)$$

The first of these is just the mass-shell condition for massless states. Having ensured that  $U$  is annihilated by  $Q'$ , we must also check to see whether it is BRST non-trivial. One way to do this is by constructing conjugate operators that have a non-vanishing inner product with  $U$ , as defined by (47). If the inner-product is non-vanishing for conjugate operators that are annihilated by  $Q'$ , then  $U$  is BRST non-trivial. Operators  $U^\dagger$  conjugate to  $U$  have the form

$$U^\dagger = f_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} \partial c c \partial \gamma \gamma \widetilde{W}^{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} \theta^\alpha e^{ip\cdot X} , \quad (57)$$

which is annihilated by  $Q'$  if

$$p^{\alpha\dot{\alpha}_1} f_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} = 0 . \quad (58)$$

It is convenient to choose a particular momentum frame in order to analyse the true physical degrees of freedom that are implied by these kinematical conditions. The null momentum vector  $p^\mu$  may, without loss of generality, be chosen to be  $p^\mu = (1, 0, 0, 1)$ . From (3), this implies that all components of  $p^{\alpha\dot{\alpha}}$  are zero except for  $p^{1\dot{1}} = \sqrt{2}$ . In this frame, the solutions to (56) and (58) are

$$h_{1\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} = 0 , \quad f_{1\dot{1}\dot{\alpha}_2\cdots\dot{\alpha}_{2s}} = 0 . \quad (59)$$

The inner product has the form  $\langle U^\dagger U \rangle = f^{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} h_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} = f^{2\dot{1}\cdots\dot{1}} h_{2\dot{1}\cdots\dot{1}}$ . Thus there is just one physical degree of freedom described by  $U$ , corresponding to the polarisation spinor component  $h_{2\dot{1}\cdots\dot{1}}$ . The other non-vanishing components of  $h_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s}}$  allowed by (59) correspond to BRST trivial states, and can be expressed back in covariant language as pure-gauge states with

$$h_{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} = p_{\alpha(\dot{\alpha}_1} \Lambda_{\dot{\alpha}_2\cdots\dot{\alpha}_{2s})} , \quad (60)$$

where  $\Lambda_{\dot{\alpha}_2\cdots\dot{\alpha}_{2s}}$  is arbitrary. We note also, for future reference, that the equation of motion for  $h^{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s}}$  in (56) is equivalent to

$$h^{\alpha\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} p_\alpha{}^{\dot{\alpha}} = 0 . \quad (61)$$

The operator  $\Psi$  in (54) is annihilated by  $Q'$  provided just that the mass-shell condition  $p^{\alpha\dot{\alpha}} p_{\alpha\dot{\alpha}} = 0$  is satisfied. To see the physical degrees of freedom, we again consider conjugate operators  $\Psi^\dagger$ , which have the form  $\Psi^\dagger = f_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} \partial c c \partial \gamma \gamma \widetilde{W}^{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} \theta^2 e^{ip\cdot X}$ . This is annihilated by  $Q'$  provided that  $p^{\alpha\dot{\alpha}_1} f_{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} = 0$ . In the special momentum frame, the solution is  $f_{1\dot{\alpha}_2\cdots\dot{\alpha}_{2s}} = 0$ . Thus the inner product is proportional to  $h^{2\cdots 2} f_{2\cdots 2}$ , so only the one component  $h^{2\cdots 2}$  describes a true physical degree of freedom. The unphysical BRST-trivial components correspond to polarisation spinors of the pure-gauge form

$$h^{\dot{\alpha}_1\cdots\dot{\alpha}_{2s}} = p^{\alpha(\dot{\alpha}_1} \Lambda_\alpha{}^{\dot{\alpha}_2\cdots\dot{\alpha}_{2s})} . \quad (62)$$

The analysis of the operators  $V$  and  $\Phi$  in (54) is precisely the same as the above analyses for  $U$  and  $\Psi$  respectively.

So far, we have concentrated on massless states in the physical spectrum. There are also massive physical states, an example being  $c e^{-\phi_1 - \phi_2} e^{ip\cdot X}$  with  $p^{\alpha\dot{\alpha}} p_{\alpha\dot{\alpha}} = -2$ , implying  $(\text{mass})^2 = 2$ . Further examples are

$$V = c \partial^{2n} \beta \cdots \partial \beta \beta (\partial^n p)^2 \cdots (\partial p)^2 p^2 e^{n(\phi_1 + \phi_2)} e^{ip\cdot X} , \quad (63)$$

where  $p^2 = p^\alpha p_\alpha$ , *etc.* These spacetime scalar states are physical for arbitrary integer  $n$ , with  $(\text{mass})^2 = 2(n+1)(2n+3)$ .

Another class of physical states in the theory is associated with infinite-dimensional representations of  $SL(2, R)_L$ . Consider, for example, the operators

$$\begin{aligned} X &= c e^{-\phi_1} e^{ip \cdot X} , \\ Y &= c \partial \gamma \gamma \theta^2 e^{-2\phi_1 - \phi_2} e^{ip \cdot X} . \end{aligned} \quad (64)$$

This is annihilated by the BRST operator provided that the mass-shell condition  $p^{\alpha\dot{\alpha}} p_{\alpha\dot{\alpha}} = 0$  is satisfied, together with the transversality condition  $p^{\alpha\dot{2}} = 0$ . This condition is not covariant with respect to  $SL(2, R)_L$ , suggesting that further terms should be added in order to construct a fully-covariant physical operator. This is analogous to viewing a physical operator built using  $W_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$  as consisting of the term involving  $W_{\dot{1} \dots \dot{1}}$  plus the remaining  $2s$  terms obtained by acting repeatedly on this highest-weight state with  $J_-$ . Thus, noting that  $e^{-2\phi_1 - \phi_2}$  is a highest-weight state,  $J_+ e^{-2\phi_1 - \phi_2} = 0$ , we may replace (64) by the  $SL(2, R)_L$  covariant operators

$$\begin{aligned} X &= \sum_{n \geq 0} h_n c \left( (J_-)^n e^{-\phi_1} \right) e^{ip \cdot X} , \\ Y &= \sum_{n \geq 0} h_n c \partial \gamma \gamma \theta^2 \left( (J_-)^n e^{-2\phi_1 - \phi_2} \right) e^{ip \cdot X} . \end{aligned} \quad (65)$$

One can easily see from the form of the generator  $J_-$  in (43) that in this case the process of repeated application of  $J_-$  will never terminate, and the sum over  $n$  will be an infinite one, corresponding to an infinite-dimensional representation of  $SL(2, R)_L$ . The physical-state conditions will now give a transversality condition on the components  $h_n$  of the polarisation tensor, rather than the non-covariant condition  $p^{\alpha\dot{2}} = 0$  that resulted when only the  $n = 0$  term was included. The occurrence of infinite-dimensional representations of  $SL(2, R)_L$  is an undesirable feature of the theory. We shall see later, the extra conditions (32) will eliminate these types of states.

## 5.2 Cohomology of the reducible system

In the previous subsections, we obtained some examples of physical states for the nilpotent operator  $Q'$ . Not all of them, however, can give rise to physical states of the fully covariant BRST operator  $Q$  according to the prescription given in (29). In this subsection, we shall examine the extra conditions (32) on these states in order to determine which states will be eliminated.

For the states of the form (29), the matter term  $\mathcal{P}_\alpha$  only acts on the operator  $V'$ , and always gives rise to a first order pole in the operator product expansion with the massless physical operators we discussed in the previous subsections. The ghost terms  $\mathcal{C}_{k+1} \mathcal{B}_k$  for  $k \geq 1$  in the extra conditions (32) will only act on the ghost for ghost operator, which has standard ghost structure, and will give rise to a first order zero in the operator product expansion. Thus these terms in (32) will annihilate the states. However, when  $k = 0$ , the ghost term  $\mathcal{C}_1 \mathcal{B}_0$  will give rise to a zeroth order pole in the operator product expansion with the  $V'$  given by either (64) or (65) or their conjugates, whilst it gives a first order zero with the  $V'$  corresponding to any of the other massless operators we discussed previously. Thus the extra conditions arising from the ghosts for ghosts eliminate the states (29) with  $V'$  given by the massless operators that are infinite-dimensional representations of  $SL(2, R)_L$ , given by (64) and (65). The massive operators (63) will also be eliminated by the extra conditions since the  $\mathcal{P}_\alpha$  term will produce

higher order poles in the operator product expansions. However this does not necessarily imply that there are no massive operators in the physical spectrum of the reducible system since we have only considered physical states with standard ghost structure for the ghosts for ghosts.

In order to compare the physical spectra of the nilpotent operator  $Q'$  of the reducible system and the BRST operator  $Q$  of the irreducible system, it is instructive to discuss the interactions. Interactions amongst the physical states that we have found so far are not easy to come by. One example that can occur is a three-point interaction  $\langle UU\Psi \rangle$  between two fermions and a boson in the scalar supermultiplet corresponding to the  $s = 0$  operators, as given in (54). The four-point amplitudes involving only the above operators are zero. This is precisely the same result we obtained for the irreducible system. For higher values of  $s$ , interactions necessarily involve tilded physical states and picture-changing operators. However all the tilded physical states have vanishing normal-ordered products with the picture-changing operators, and thus there are no interactions among these states. Thus all the physical operators with higher values of  $s$  decouple from the theory.

The physical operators under  $Q'$  that carry infinite-dimensional representations of  $SL(2, R)_L$ , however can have interactions with the  $s = 0$  spinor operators. For example, we can build a non-vanishing four-point amplitude  $\langle U V X X \rangle$  without the need for the picture changing, where  $U, V$  are given by (54) with  $s = 0$ , and  $X$  is given by (65). This would be inconsistent with the results of the irreducible system, where no non-vanishing four-point amplitude can be built from purely massless operators. However, this apparent discrepancy is overcome by the introduction of the ghosts for ghosts, since as we discussed above, these operators will be eliminated by the extra conditions in (32).

## 6 Discussion and conclusions

In this paper, we have constructed a superstring theory in four-dimensional spacetime with  $(2, 2)$  signature, using the Berkovits' approach of augmenting the spacetime supercoordinates by the conjugate momenta for the fermionic variables [2]. The form of the theory, and its local worldsheet symmetries, was motivated by Siegel's proposal [6] for a set of constraints that could give rise to self-dual super Yang-Mills theory or supergravity in  $2 + 2$  dimensions. In the theory that we have considered,  $N = 1$  spacetime supersymmetry is manifest in the formulation, as is the right-handed  $SL(2, R)$  factor of the  $SO(2, 2) \equiv SL(2, R)_L \times SL(2, R)_R$  Lorentz group.

The constraints that we have used are a subset of Siegel's constraints [6] that form a closed algebra under commutation. They are manifestly  $SO(2, 2)$  covariant; however, they suffer from the fact that they are reducible, with an infinite order of reducibility. This implies that the fully covariant BRST treatment requires infinitely many ghosts for ghosts. Nevertheless, we can construct the nilpotent operator for the reducible system. We studied the cohomology of this nilpotent operator. It gives rise to a theory with an infinite number of massless states with arbitrary spin. This feature can be attributed to the fact that the fermionic constraint carries an  $SL(2, R)_L$  spinor index, leading to the existence of ghost vacua with arbitrary spin  $s$  under  $SL(2, R)_L$ . However these higher-spin massless states are decoupled from the  $s = 0$  states.

One way to overcome the reducibility problem is by choosing an irreducible subset of constraints, at the cost of breaking the spacetime Lorentz covariance. Since the irreducible constraints also have critical central charge in our case, we were able to construct the BRST operator and study its cohomology. We showed that the theory also maintains the manifest  $N = 1$  su-



persymmetry. The physical spectrum with standard ghost structure includes a massless bosonic operator and its superpartner. Their interactions comprise only a three-point amplitude with one insertion of the boson and two insertions of the fermion. This pattern of interactions is precisely the same as that of the  $s = 0$  massless states of the reducible system.

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